Midterm sample practice exam problems

1. Calculate the inertia tensor of an ellipsoid shaped thin wire.

2. Four equal, uniform rods of mass m and length 2a are hinged together to form a rhombus ABCD. The point A is fixed, while C lies directly beneath it and is free to slide up and down. The whole system can rotate around the vertical. Let θ be the angle that AB makes with the vertical, and ω be the angular velocity around the vertical. Find the Lagrangian for this system and show that there are two conserved constants of motion.

3. The set of equations

$$q_1 = (2Q_1/\lambda_1)^{1/2}\cos P_1 + (2Q_2/\lambda_2)^{1/2}\cos P_2$$

$$q_2 = -(2Q_1/\lambda_1)^{1/2}\cos P_1 + (2Q_2/\lambda_2)^{1/2}\cos P_2$$

$$p_1 = (Q_1\lambda_1/2)^{1/2}\sin P_1 + (Q_2\lambda_2/2)^{1/2}\cos P_2$$

$$p_2 = -(Q_1\lambda_1/2)^{1/2}\sin P_1 + (Q_2\lambda_2/2)^{1/2}\cos P_2$$

defines a canonical transformation for the dynamical system with the Hamiltonian function

$$H = p_1^2 + p_2^2 + \frac{1}{8}\lambda_1^2(q_1 - q_2)^2 + \frac{1}{8}\lambda_2^2(q_1 + q_2)^2$$

(a) Find the transformed Hamilton function H'.
(b) Solve Hamilton's equations of motion for the transformed Hamiltonian H', and hence for the original system.
(c) Explain the physical meaning of $\lambda_1, \lambda_2, P_1, P_2, Q_1, Q_2$.

4. Solve the Hamilton-Jacobi differential equation for a particle moving in a uniform gravitational field.

5. The kinetic and potential energy of there coupled oscillator, in terms of small displacements $\eta_i(t)$ from the equilibrium is given by

$$T = \frac{1}{2}m\sum_{i=1}^{3}\dot{\eta}_i^2 \quad (1)$$

$$V = \frac{1}{2}k\sum_{i=1}^{3}\eta_i^2 + k'(\eta_1\eta_2 + \eta_2\eta_3) \tag{2}$$

The initial condition at $t = 0$ is

$$(\eta_1(0), \eta_2(0), \eta_3(0)) = (x_0, 0, 0) \qquad (\dot{\eta}_1(0), \dot{\eta}_2(0), \dot{\eta}_3(0)) = (0, 0, v_0) \tag{3}$$

a) Calculate the normal frequencies ω_1, ω_2 and ω_3.

b) Calculate the corresponding orthonormalized eigenvectors.

c) Write down the solution for $\eta(t)$ using the initial conditions.