XV. EXAMPLES

A. Rotation around a fixed axis: Compound pendulum

If one axis is fixed then we can choose the z axis of the body frame as the axis of rotation, $\omega_1 = \omega_2 = 0$ and the pendulum position measured from the vertical direction is defined by ϕ

$$\dot{L}_3 = \Gamma_3 \qquad L_3 = I_3\omega_3 = I_3\dot{\phi}$$

The torque is due to the gravitational force

$$\mathbf{\Gamma} = \int \rho(\mathbf{r})\mathbf{r} \times \mathbf{g} = M\mathbf{R} \times \mathbf{g}$$

where M is the total mass, $\mathbf{R} = (R\cos\phi, R\sin\phi, 0)$ is the center of mass $\mathbf{g} = (g, 0, 0)$, (z is the rotation axis and the pendulum moves in the xy plane, x directed downward) that is

$$\Gamma_3 = -Mgl\sin\phi.$$

The equation of motion is

$$I_3\ddot{\phi} = -Mgl\sin\phi.$$

B. Force-Free Symmetric top

The Euler equations for a force free symmetric top ($\theta_1 = \theta_2 \neq \theta_3$)

$$\begin{aligned} \Theta_1\dot{\omega}_1 &= \omega_2\omega_3(\theta_1 - \theta_3) \\ \Theta_1\dot{\omega}_2 &= \omega_3\omega_1(\theta_3 - \theta_1) \\ \Theta_3\dot{\omega}_3 &= 0 \end{aligned}$$

From the last equation (if $\theta_3 \neq 0$) ω_3=constant Defining

$$\omega_p = \omega_3\left(\frac{\theta_3}{\theta_1} - 1\right)$$

the Euler equation becomes

$$\begin{aligned} \dot{\omega}_1 &= -\omega_p\omega_2 \\ \dot{\omega}_2 &= \omega_p\omega_1 \end{aligned}.$$

By substituting one of the equation into the other one one gets a second order linear differential equation, just like the harmonic oscillator

$$\ddot{\omega}_1 = -\omega_p^2\omega_1$$

The general solution is

$$\omega_1(t) = \omega_0 \cos(\omega_p t + \phi_0) \qquad \omega_2(t) = \omega_0 \sin(\omega_p t + \phi_0)$$

where ω_0 and ϕ_0 are defined by the initial conditions.

The magnitude of the angular velocity

$$\omega = \sqrt{\omega_1(t)^2 + \omega_2(t)^2 + \omega_3^2} = \text{constant.}$$

Since ω_3 and $\sqrt{\omega_1(t)^2 + \omega_2(t)^2} = \omega_0$ =constant, the angle between the $\mathbf{f}_3$ axis and $\boldsymbol{\omega}$ is also constant. That means that $\boldsymbol{\omega}$ is precessing around $\mathbf{f}_3$ with angular frequency ω_p. Example: Earth!

Note: $\theta_3 > \theta_1$: oblate

Note: $\theta_3 < \theta_1$: prolate (cigar)

Careful: This description is valid in body frame. In the inertial frame everything is more complicated!

C. Rotational kinetic energy of a symmetric top

$$T_{rot} = \frac{1}{2}\left(\theta_1(\omega_1^2 + \omega_2^2) + \theta_3 \omega_3^2\right)$$

or, explicitly, using the Euler angles

$$T_{rot} = \frac{1}{2}\left(\theta_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \theta_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\right)$$

For the force-free case the Lagrangian

$$L(\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}) = T_{rot} = \frac{1}{2}\left(\theta_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \theta_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\right)$$

where ϕ and ψ are cyclic variables, therefore the canonila angular momenta are conserved:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \theta_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta + \theta_1 \sin^2\theta \dot{\phi}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = \theta_3(\dot{\psi} + \dot{\phi}\cos\theta) = \theta_3 \omega_3$$

Expressing $\dot{\phi}$ and $\dot{\psi}$ from these equations, the kinetic energy becomes:

$$T_{rot} = \frac{1}{2}\left(\theta_1 \dot{\theta}^2 + \theta_3 \omega_3^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{\theta_1 \sin^2\theta}\right)$$

This means that the force free motion of the symmetric top can be described in term of the Euler-Lagrange equation for the Euler angle θ

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \rightarrow \theta_1 \ddot{\theta} = -\frac{(p_\phi - p_\psi \cos\theta)^2}{\theta_1 \sin\theta} \frac{(p_\psi - p_\phi \cos\theta)^2}{\sin^2\theta}$$

Once $\theta(t)$ is solved, $\phi(t)$ and $\psi(t)$ can be calculated using the equations above.

D. Symmetric top with one fixed point

Now consider the case of symmetric top in gravitational field. The potential energy is

$$V(\theta) = Mgh\cos\theta \tag{1}$$

The Lagrangian

$$L(\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}) = \frac{1}{2}\left(\theta_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \theta_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\right) - Mgh\cos\theta$$

Similarly to the previous case, ϕ and θ are cyclic:

$$L = \frac{1}{2}\left(\theta_1(\dot{\theta}^2 + \theta_3 \omega_3^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{\theta_1 \sin^2\theta}\right) - Mgh\cos\theta$$

The constants of motion is the energy

$$E = \frac{1}{2}\left(\theta_1(\dot{\theta}^2 + \theta_3 \omega_3^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{\theta_1 \sin^2\theta}\right) + Mgh\cos\theta$$

and p_ϕ and $p_\psi = \theta_3 \omega_3$. This means that

$$E' = E - \frac{1}{2}\theta_3 \omega_3^2$$

is also constant. We can define an “effective” potential

$$U(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2\theta_1 \sin^2\theta} + Mgh\cos\theta$$

and we have

$$E' = \frac{1}{2}\theta_1 \dot{\theta}^2 + U(\theta)$$

which can be solved as

$$t(\theta) \pm \int \frac{d\theta}{\sqrt{(2/\theta_1)(E' - U(\theta)}}$$

The shape of the effective potential determined the motion in the θ angle. This potential is a parabolic barrier and keeps top moving between two angles.