XIII. DAMPED OSCILLATIONS, RESONANCES

A. Linear differential equations

The equation

$$\sum_{k=0}^{n} g_k(t) y^{(k)}(t) = f(t)$$

where $g_k(t)$ and $f(t)$ are given functions is called n-th order linear differential equation. If $f(t) = 0$ then the equation is called homogeneous otherwise it is inhomogeneous differential equation. We are looking for the solution of this equation which satisfies the

$$y^{(k)}(t_0) = y_0^{(k)} \qquad (k = 0, \ldots, n-1)$$

initial conditions.

The homogeneous equation has n linearly independent solutions $\varphi_i(t)$ $(i = 1, \ldots, n$. The solution of the inhomogeneous equation can be written in the form

$$y(t) = \sum_{k=0}^{n} a_k \varphi_k(t) + \phi$$

where a_k are constants and ϕ is a particular solution of the inhomogenous equation. This is called general solution.

If the coefficients are constant $(g_k(t) = c_k)$, then the solution can be written as

$$y(t) = A\mathrm{e}^{\lambda t}$$

substituting the into the homogeneous equation we get the

$$\sum_{k=0}^{n} c_k \lambda^k = 0$$

characteristic equation.

B. Linear oscillator

Equation of motion

$$m\ddot{\mathbf{r}} + D\mathbf{r} = 0$$

Characteristic equation

$$m\lambda^2 + D = 0 \qquad \rightarrow \lambda_{1,2} = \pm i\omega \quad (\omega = \sqrt{D/m})$$

General solution

$$\mathbf{r}(t) = \mathbf{A_1}\mathrm{e}^{i\omega t} + \mathbf{A_2}\mathrm{e}^{i\omega t}$$

The initial condition

$$\mathbf{r}(0) = \mathbf{r}_0 = \mathbf{A_1} + \mathbf{A_2}$$

$$\mathbf{v}(0) = \mathbf{v}_0 = i\omega(\mathbf{A_1} - \mathbf{A_2}).$$

from these equations:

$$\mathbf{A_1} = \frac{1}{2}(\mathbf{r}_0 + \mathbf{v}_0/(i\omega)) \qquad \mathbf{A_2} = \frac{1}{2}(\mathbf{r}_0 - \mathbf{v}_0/(i\omega))$$

and then

$$\mathbf{r}(t) = \mathbf{r}_0\cos(\omega t) + \mathbf{v}_0\omega\sin(\omega t)$$

C. Damped oscillator

Equation of motion

$$m\ddot{\mathbf{r}} + \beta\dot{\mathbf{r}} + D\mathbf{r} = 0$$

The characteristic equation (the roots of $ax^2 + bx + c = 0$ are $(-b \pm \sqrt{b^2 - 4ac})/(2a)$

$$m\lambda^2 + \beta\lambda + D = 0 \qquad \to \lambda_{1,2} = -\kappa \pm \sigma$$

where

$$\kappa = \frac{\beta}{2m} \qquad \sigma = \sqrt{\kappa^2 - \omega^2}$$

If $\sigma \neq 0$ then there are two independent solutions

$$A_1\mathrm{e}^{\lambda_1 t} \qquad A_2\mathrm{e}^{\lambda_2 t}$$

If $\sigma = 0$ then there are two independent solutions

$$A_1\mathrm{e}^{-\kappa t} \qquad tA_2\mathrm{e}^{-\kappa t}$$

Using the initial velocity and position (note: $\mathrm{sh}x = (\mathrm{e}^x - \mathrm{e}^{-x})/2$ and $\mathrm{ch}x = (\mathrm{e}^x + \mathrm{e}^{-x})/2$)

$$\mathbf{r}(t) = \begin{cases} \mathrm{e}^{-\kappa t}(\mathbf{r}_0(\mathrm{ch}(\sigma t) + \frac{\kappa}{\sigma}\mathrm{sh}(\sigma t)) + \frac{\mathbf{v}_0}{\sigma}\mathrm{sh}(\sigma t)) & \text{for } \sigma \neq 0 \\ \mathrm{e}^{-\kappa t}(\mathbf{r}_0(1 + \kappa t) + \mathbf{v}_0 t) & \text{for } \sigma = 0 \end{cases},$$

defining $\omega_c = \sqrt{|\omega^2 - \kappa^2|}$, there are two cases, $\sigma = i\omega_c$ or $\sigma = \omega_c$

$$\mathbf{r}(t) = \begin{cases} \mathrm{e}^{-\kappa t}(\mathbf{r}_0(\cos(\sigma t) + \frac{\kappa}{\sigma}\sin(\sigma t)) + \frac{\mathbf{v}_0}{\sigma}\sin(\sigma t)) & \text{for } \omega^2 > \kappa^2 \\ \mathrm{e}^{-\kappa t}(\mathbf{r}_0(1 + \kappa t) + \mathbf{v}_0 t) & \text{for } \omega^2 = \kappa^2 \\ \mathrm{e}^{-\kappa t}(\mathbf{r}_0(\mathrm{ch}(\sigma t) + \frac{\kappa}{\sigma}\mathrm{sh}(\sigma t)) + \frac{\mathbf{v}_0}{\sigma}\mathrm{sh}(\sigma t)) & \text{for } \omega^2 < \kappa^2 \end{cases}$$

Note that the due to the $\mathrm{e}^{-\kappa t}$ damping factor

$$\mathbf{r}(t \to \infty) = 0$$

For $\omega^2 > \kappa^2$ we have

$$\mathbf{r}(t) = (\mathbf{r}_0\sin(\omega_c t + \delta) + \mathbf{v}_0\omega_c\sin(\omega_c t))A\mathrm{e}^{-\kappa t}$$

where

$$A = \omega/\omega_c \qquad \mathrm{tg}\delta = \omega_c/\kappa$$

For $\omega^2 < \kappa^2$ we have

$$\mathbf{r}(t) = (\mathbf{r}_0\mathrm{sh}(\omega_c t + \delta) + \mathbf{v}_0\omega_c\mathrm{sh}(\omega_c t))A\mathrm{e}^{-\kappa t}$$

where

$$A = \omega/\omega_c \qquad \mathrm{th}\delta = \omega_c/\kappa < 1$$

Note: δ phase difference, damped oscillations.

D. Forced oscillator

Equation of motion

$$m\ddot{\mathbf{r}} + \beta\dot{\mathbf{r}} + D\mathbf{r} = \mathbf{F}_0\cos(\Omega t)$$

It is easier to solve the complex equations:

$$m\ddot{\mathbf{r}} + \beta\dot{\mathbf{r}} + D\mathbf{r} = \mathbf{F}_0\mathrm{e}^{i\Omega t}$$

(the two equation is identical and equivalent assuming that $\mathbf{F}_0$ has a vanishing imaginary part). The solution is of the form

$$\mathbf{r}(t) = \mathbf{r}_h(t) + \mathbf{r}_p(t)$$

where $\mathbf{r}_h(t)$ is the solution of the homogeneous equation and $\mathbf{r}_p(t)$ is a particular solution. The solution of the homogeneous equation is given in the preceeding section, now we are looking for the particular solution in the form

$$\mathbf{r}_p(t) = \mathbf{c}\mathrm{e}^{i\Omega t}$$

and by substituting this into the differential equation

$$(-\Omega^2 + 2i\Omega\kappa + \omega^2)\mathbf{c} = \mathbf{F}_0/m.$$

Defining

$$\Lambda\mathrm{e}^{i\delta} = -\Omega^2 + 2i\Omega\kappa + \omega^2$$

that is

$$\Lambda = \sqrt{(\omega^2 - \Omega^2)^2 + 4\kappa^2\Omega^2} \qquad \mathrm{tg}\delta = \frac{2\kappa\Omega}{\omega^2 - \Omega^2}$$

(Note: Polar representation of complex numbers: $r\mathrm{e}^{i\phi} = u + iv$ then $r^2 = u^2 + v^2$ and $\mathrm{tg}\phi = v/u$.) Therefore

$$\mathbf{c} = \frac{\mathbf{F}_0}{m\Lambda}\mathrm{e}^{-i\delta}$$

and

$$\mathbf{r}_p = Re\left[\frac{\mathbf{F}_0}{m\Lambda}\mathrm{e}^{i(\Omega t - \delta)}\right] = \frac{\mathbf{F}_0}{m\Lambda}\cos(\Omega t - \delta)$$

or

$$\mathbf{r}_p = \mathbf{A}\cos(\Omega t - \delta) \qquad \mathbf{A} = \frac{\mathbf{F}_0}{m\sqrt{(\omega^2 - \Omega^2)^2 + 4\kappa^2\Omega^2}}$$

Note: Both the amplitude $\mathbf{A}$ and the phase δ depends on the driving frequency Ω! The maximum of the amplitude is at

$$\frac{\partial \mathbf{A}(\Omega)}{\partial \Omega} = 0 \qquad \to \Omega_{max} = \sqrt{\omega^2 - 2\kappa^2}$$

where

$$\mathbf{A}_{\max} = \mathbf{A}(\Omega_{max}) = \frac{\mathbf{F}_0}{m\kappa\sqrt{\omega^2 - \kappa^2}}$$