XII. SMALL OSCILLATIONS: EXAMPLES

A. Pendulum

$$L = T - V = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi$$

Equilibrium:

$$\frac{\partial V}{\partial \phi} = mgl\sin\phi = 0$$

that is $\phi_0 = 0$ the equilibrium position.

$$T_{11} = m \qquad V_{11} = \left(\frac{\partial^2 V}{\partial \phi^2}\right)_{\phi_0} = (mgl\cos\phi)_{\phi_0} = mgl$$

The Lagrange equation for small $\eta = l\phi$ displacement:

$$T_{11}\ddot{\eta} + V_{11}\eta = m\ddot{\eta} + m\frac{g}{l}\eta = 0$$

Substituting

$$\eta = A\cos(\omega t) + B\sin(\omega t)$$

the normal frequency is

$$\omega = \sqrt{\frac{g}{l}}$$

B. Pendulum with an elastic string

The Lagrangian is

$$L = T - V = \frac{m}{2}(\dot{x}^2 + x^2\dot{\phi}^2) + mgx\cos\phi - \frac{1}{2}D(x - l_0)^2$$

where the generalized coordinates

$$\underline{q} = (x, \phi)$$

The equilibrium that is

$$\underline{q}_0 = (l_0 + mg/D, 0)$$

The kinetic energy matrix

$$T = \begin{pmatrix} m & 0 \\ 0 & mx^2 \end{pmatrix}_{\underline{q}_0} = \begin{pmatrix} m & 0 \\ 0 & m(l_0 + mg/D)^2 \end{pmatrix}$$

The potential energy matrix

$$V = \begin{pmatrix} D & mg\sin\phi \\ mg\sin\phi & mgx\cos\phi \end{pmatrix}_{\underline{q}_0} = \begin{pmatrix} D & 0 \\ 0 & mg(l_0 + mg/D) \end{pmatrix}$$

(Note that this is a very special case: Both T and V are diagonal, we do not have to diagonalize them). The normal frequencies can be calculated from

$$\det(\underline{V} - \omega^2 \underline{T}) = \begin{pmatrix} D - \omega^2 m & 0 \\ 0 & mg(l_0 + mg/D) - \omega^2 m(l_0 + mg/D)^2 \end{pmatrix} = 0$$

and the two normalfrequencies are:

$$\omega_1 = \sqrt{D/m} \qquad \omega_2 = \sqrt{\frac{g}{l_0 + mg/D}}$$

C. Double pendulum

The Lagrangian:

$$L = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + (m_1 + m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

the generalized coordinates

$$\underline{q} = (\phi_1, \phi_2)$$

Equilibrium:

$$\frac{\partial V}{\partial \phi_1} = (m_1 + m_2) g l_1 \sin\phi_1 \qquad \frac{\partial V}{\partial \phi_2} = m_2 g l_2 \sin\phi_2$$

that is

$$\underline{q} = (0,0); (0,\pi); (\pi,0); (\pi,\pi)$$

(The last three are not too meaningful). The kinetic energy matrix

$$T = \begin{pmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m l_2^2 2 \end{pmatrix}$$

The potential energy matrix

$$V = \begin{pmatrix} (m_1 + m_2) g l_1 \cos\phi & 0 \\ 0 & m_2 g l_2 \cos\phi_2 \end{pmatrix}_{\underline{q}_0} = \begin{pmatrix} (m_1 + m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix}$$

From

$$\det(\underline{V} - \omega^2 \underline{T}) = 0$$

we obtain the eigenfrequencies

$$\omega_{1,2} = \sqrt{\frac{m_1 + m_2}{m_1} g \frac{l_1 + l_2}{2l_1 l_2} \left(1 \pm \sqrt{1 - \frac{m_1}{m_1 + m_2} \frac{4l_1 l_2}{(l_1 + l_2)^2}}\right)}.$$

D. Coupled pendulums

Two pendulums coupled with a spring, the first pendulum is at position θ_1 the second is at θ_2. For small displacements:

$$\mathbf{r}_1 = (l\sin\theta_1, -l\cos\theta_1, 0) \qquad \mathbf{r}_2 = (d_0 + l\sin\theta_2, -l\cos\theta_2, 0)$$

The change in the length of the spring is

$$d - d_0 = l\sin\theta_2 - l\sin\theta_1$$

The potential energies

$$V_1 = -mgl\cos\theta_1 \qquad V_2 = -mgl\cos\theta_2,$$

the potential energy of the spring is

$$V_s = \frac{1}{2} k (d - d_0)^2,$$

The potential energy of the system is

$$V = V_1 + V_2 + V_s = -mgl\cos\theta_1 - mgl\cos\theta_2 + \frac{1}{2} k (d - d_0)^2$$

In equilibrium $\phi_1 = \phi_2 = 0$. Denoting the small displacements by

$$\eta_1 = l\phi_1 \qquad \eta_2 = l\phi_2$$

we have

$$l\sin\phi_1 \approx \eta_1 \qquad l\sin\phi_2 \approx \eta_2,$$

and

$$l\cos\phi_1 \approx 1 - \frac{1}{2}\eta_1^2 \qquad l\cos\phi_2 \approx 1 - \frac{1}{2}\eta_2^2,$$

(where we had to go up to second order because we need the second derivatives of the potential energy matrix). By using these equations, the potential energy is equal to

$$V = \frac{1}{2}\frac{mg}{l}(\eta_1^2 + \eta_2^2) + \frac{1}{2}k(\eta_1 - \eta_2)^2 - 2mgl$$

The kinetic energy

$$T = \frac{1}{2}m(l^2\dot{\phi}_1^2 + l^2\dot{\phi}_2^2) = \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2)$$

The potential energy matrix is

$$V_{jk} = \left(\frac{\partial^2 V}{\partial\eta_j\partial\eta_k}\right)_{\eta_0}$$

$$V = \begin{pmatrix} k + mg/l & -k \\ -k & k + mg/l \end{pmatrix}.$$

The kinetic energy matrix is

$$T = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}.$$

The equation of motion:

$$\sum_{k=1}^{2}(T_{j,k}\ddot{\eta}_k - V_{j,k}\eta_k) = 0 \quad (j = 1, \ldots, s)$$

Assuming the solutions in the form

$$\eta_i = \rho_i \exp(i\omega t)$$

the equation of motion becomes

$$(\underline{V} - \omega^2\underline{T})\underline{\rho} = 0.$$

The determinant of $\underline{V} - \omega^2\underline{T}$ is

$$\det(\underline{V} - \omega^2\underline{T}) = m^2\omega^4 - 2m\omega^2(k + mg/l) + (k + mg/l)^2 - k^2 = 0.$$

The roots of this (quadratic) equation are given by

$$\omega_1 = \sqrt{g/l} \quad \omega_2 = \sqrt{g/l + 2k/m}$$

(The first frequency is that of a single pendulum). By substituting these eigen frequancies in to the equations of motion we can solve the equations for ρ_i. From

$$(\underline{V} - \omega_i^2\underline{T})\underline{\rho}_i = \begin{pmatrix} mg/l + k - m\omega_i^2 & -l \\ -k & mg/l + k - m\omega_i^2 \end{pmatrix}\begin{pmatrix} \rho_{i1} \\ \rho_{i2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we have

$$\underline{\rho}_1 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \underline{\rho}_2 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Using these eigenvectors we can construct a matrix

$$\underline{X} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

This matrix diagonalizes both $\underline{V}$ and $\underline{T}$:

$$\underline{X}^T \underline{V}\ \underline{X} = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}, \tag{1}$$

$$\underline{X}^T \underline{T}\ \underline{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

The normal coordinates:

$$\underline{\xi} = \underline{X}^T\ \underline{T}\underline{\eta} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \sqrt{\frac{m}{2}} \begin{pmatrix} \eta_1 + \eta_2 \\ \eta_2 - \eta_2 \end{pmatrix} \tag{3}$$

E. Longitudinal motion of N particles connected with springs

We have N particles on a line connected by springs. The first and last particle connected to fixed endpoints through springs. The displacement of particles from the equilibrium position is η_i ($i = 1, \ldots, N$. The Lagrangian

$$L = \frac{1}{2} m \sum_{i=1}^{N} \dot{\eta}_i - \frac{1}{2} D \sum_{i=0}^{N} (\eta_{i+1} - \eta_i)^2$$

The endpoints are fixed: $\eta_0 = 0$ and $\eta_{N+1} = 0$. The equations of motions

$$m\ddot{\eta}_i - k(\eta_{i+1} - \eta_i) + k(\eta_i - \eta_{i-1}) = 0$$

Note: Coupled to the neighbors only!

F. Transverse motion of N identical masses

N identical masses equally spaced on a massless sting. If a mass is out of equilibrium by an infinitesimally small x displacement perpendicular to the relaxed string then the tension force acting on that mass is

$$F(x) = -\tau \sin\alpha \approx -\tau \frac{x}{\sqrt{x^2 + a^2}}$$

where a is the distance between the masses and α is the angle of the string measured from the relaxed position. The potential energy

$$V(x) = -\int_0^x F(x')dx' = \tau \int_0^x \frac{x}{\sqrt{x^2 + a^2}} dx' = \tau(\sqrt{x^2 + a^2} - a)$$

For small x

$$\sqrt{x^2 + a^2} = a\sqrt{1 + (x/a)^2} \approx a(1 + \frac{1}{2}\frac{x^2}{a^2})$$

$$U(x) = \sqrt{x^2 + a^2} = \frac{1}{2}\frac{\tau}{a}x^2$$

For the N mass system

$$U(x_1, \ldots, x_N) = \frac{1}{2}\frac{\tau}{a}\sum_{i=0}^{N}(x_{i+1} - x_i)^2$$

Analogue to the previous example.