VIII. INTEGRATION OF THE EQUATIONS OF MOTION

A. On dimensional motion

In one dimension:

$$L = \frac{1}{2}m\dot{x}^2 - V(x),$$

from the conservation of energy

$$E = \frac{1}{2}m\dot{x}^2 + V(x) \rightarrow \quad \frac{dx}{dt} = \sqrt{2(E - V(x)/m}.$$

The right hand side does not depend on time so it can be readily integrated

$$t = \sqrt{m/2}\int \frac{dx}{\sqrt{(E - V(x)}} + C.$$

There are two arbitrary constant in the solution: E and C. Important! The values of these constant can be calculated from the initial conditions.

The points where

$$E - V(x) = 0$$

are called turning points (the velocity is zero).

B. Example

Assume that a pendulum starts from the initial position ϕ_0. The potential is

$$V(\phi) = -mgl\cos\phi,$$

the total energy of the pendulum is

$$E = -mgl\cos\phi_0 = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

The time of motion between $\phi = 0$ and $\phi = \phi_0$ is

$$T = \sqrt{l/2g}\int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

(This integral can be calculated by using the *complete elliptic integral of the first kind*).

IX. MOTION IN A CENTRAL FIELD

Central field: the potential $V(r)$ depends only on a distance r from a fixed point. The force acting on a particle

$$F = -\nabla V(r) = -D\frac{\mathbf{r}}{|\mathbf{r}|}$$

The torque

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = 0$$

therefore the angular momentum is constant:

$$\mathbf{L(t)} = \mathbf{L}_0$$

The particle moves in a plane perpendicular to $\mathbf{L}_0$ (which determined by the initial conditions):

$$\mathbf{r} \cdot \mathbf{L}_0 = \mathbf{r} \cdot (\mathbf{r} \times \mathbf{p}) = 0$$

We can describe the motion of this particle using plane polar coordinates:

$$x = r\text{cos}\phi \qquad y = r\text{sin}\phi$$

Reduced mass:

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|)$$

We can introduce the relative and center of mass coordinates

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \qquad \mathbf{R} = \frac{m_1\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 + m_2}.$$

We can express $\mathbf{r}_1$ and $\mathbf{r_2}$ by these *relative* coordinates

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} + \mathbf{R} \qquad \mathbf{r}_2 = -\frac{m_1\mathbf{r}}{m_1 + m_2} + \mathbf{R}.$$

Using these relative coordinates, the Lagrangian reads as

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(|\mathbf{r}|),$$

where

$$m = \frac{m_1m_2}{m_1 + m_2}$$

is the reduced mass. Using plane polar coordinates:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r).$$

Cyclic coordinates: If a generalised coordinate q_i is missing from the Lagrangian then that coordinate is called cyclic. For this coordinate

$$\frac{\partial L}{\partial q_i} = 0,$$

and the Lagrange equation takes the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0,$$

so the corresponding generalized momentum p_i is an integral (constant) of the motion.

A. Example

In a central field the Lagrangian does not explicitly depend of ϕ therefore

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}_i} = mr^2\dot{\phi} = l = constant$$

This can be written in a different form. $A = r(rd\phi)$ is the area of the angular interval (sector) when the particle moves from ϕ to $\phi + d\phi$.

$$\dot{A} = \frac{l}{2m} = constant$$

This is Kepler's second law: The sectorial velocity is constant.

X. MOTION IN A GRAVITATIONAL FIELD

The potential in the gravitational field is

$$V(r) = \frac{\alpha}{r} \qquad \alpha = \gamma m_1 m_2$$

while the force is

$$\mathbf{F} = -\nabla V(r) = -\frac{\alpha}{r^3}\mathbf{r}$$

In this case, besides the energy and the angular momentum $\mathbf{L}$, there is a third constant of motion the Lenz-Runge (or excentricity) vector:

$$\epsilon = \frac{1}{\alpha}\mathbf{v} \times \mathbf{L} - \frac{\mathbf{r}}{r}$$

This vector is very useful because (as all integral of motion) can be calculated from the initial conditions and completely determines the trajectory of the motion.

Proof:

$$\dot{\epsilon} = \frac{1}{\alpha}\dot{\mathbf{v}} \times \mathbf{L} + \frac{1}{\alpha}\mathbf{v} \times \dot{\mathbf{L}} - \frac{d}{dt}\frac{\mathbf{r}}{r}$$

Here $\dot{\mathbf{L}} = 0$ ($\mathbf{L} = \mathbf{L}_0$) and from the equation of motion

$$\dot{\mathbf{v}} = -\frac{\alpha}{m|\mathbf{r}|^3}\mathbf{r}.$$

The time derivative of a unit vector:

$$\frac{d}{dt}\frac{\mathbf{r}}{r} = \frac{\dot{\mathbf{r}}|\mathbf{r}| - \left(\frac{d}{dt}|\mathbf{r}|\right)\mathbf{r}}{|\mathbf{r}|^2} = \frac{\dot{\mathbf{r}}\mathbf{r}^2 - \mathbf{r}(\dot{\mathbf{r}}\mathbf{r})}{|\mathbf{r}|^3} = \frac{\mathbf{r} \times (\dot{\mathbf{r}} \times \mathbf{r})}{|\mathbf{r}|^3} = -\frac{\mathbf{r} \times (\mathbf{r} \times \mathbf{p})}{m|\mathbf{r}|^3} = \frac{\mathbf{r} \times \mathbf{L}}{m|\mathbf{r}|^3}$$

where

$$\frac{d}{dt}|\mathbf{r}| = \frac{\mathbf{r}\dot{\mathbf{r}}}{|\mathbf{r}|}$$

and the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

is used. Therefore

$$\dot{\epsilon} = 0,$$

ϵ is a constant of motion.

The excentriticy vector is perpendicular to $\mathbf{L}$:

$$\epsilon \cdot \mathbf{L} = 0.$$

The trajectory of the motion can be derived from

$$\epsilon \cdot \mathbf{r} = \frac{1}{\alpha}\mathbf{r} \cdot (\mathbf{v} \times \mathbf{L}) - r = \frac{1}{\alpha}\mathbf{L} \cdot (\mathbf{r} \times \mathbf{v}) - r = \frac{1}{m\alpha}\mathbf{L}^2 - r$$

by using $\mathbf{r}\epsilon = r|\epsilon|\cos\phi$ and introducing $p = \frac{\mathbf{L}_0^2}{\alpha m}$ we have

$$r = \frac{p}{1 + |\epsilon|\cos\phi}$$

This equation defines the trajectory of the motion. There are four cases:

$\epsilon > 1$ *hiperbolic*

$\epsilon = 1$ *parabolic*

$\epsilon < 1$ *elipsoid*

$\epsilon = 0$ *circular*

The direction of ϵ defines the direction of the axis of the parabola/hiperbola. . . .