III. CONSTRAINED MOTION

A. Holonomic constraints:

In a N particle system, the i-th particle is characterized by the coordinates $(\mathbf{r}_{i1}, \mathbf{r}_{i2}, \mathbf{r}_{i3})$. In case of a constrained motion these coordinates are not independent of each other, but there are some relations between certain coordinates:

$$f_i(r_{11}, ...r_{N3}) = c_i \qquad (i = 1, ..., k)$$

These k relations are called *holonomic constraints* (relation between coordinates and time). <u>Generalized coordinates</u> The $s = 3N - k$ *independent* q_i coordinates which completely determines the motion of the particles called *generalized coordinates*.

B. Example: Motion on a plane

Assume that a particle is constrained to move of a plane. Let the normal vector of the plane (a unit vector which is perpendicular to the plane) be denoted by $\mathbf{n}$ ($|\mathbf{n}|$=1) and let the position vector of the particle be $\mathbf{r}$. The holonomic constraint in this case is given by

$$\mathbf{r} \cdot \mathbf{n} = h \quad (f(r_1, r_2, r_3) = \mathbf{r} \cdot \mathbf{n})$$

In this case $k = 1$ and we can choose the generalized coordinates as

$$q_1 = x \quad q_2 = y$$

and z is not independent as it can be expressed as

$$z = (h - n_1 q_1 - n_2 q_2)/n_3.$$

If the motion is constrained to the $x - y$ plane then $\mathbf{n} = (0, 0, 1)$ then

$$\mathbf{r} \cdot \mathbf{n} = r_3 = h$$

C. Example: Pendulum

If a particle attached to a string of length l then its motion is constrained by the holonomic constraint:

$$|\mathbf{r}| = l \quad (f(r_1, r_2, r_3) = |\mathbf{r}|)$$

Here $k = 1$ and we can choose

$$q_1 = x \qquad q_2 = y$$

and z is constrained to be

$$z = \sqrt{l - q_1^2 - q_2^2}$$

IV. CALCULUS OF VARIATIONS

A. Functionals

Let x be an independent variable (e.g. time) defined in the $[x_1, x_2]$ interval and let $y(x)$ be a differentiable function of x in this interval. Suppose that there is a functional (function of functions)

$$\phi(y(x), y'(x), x) \qquad y'(x) = dy/dx$$

and we want to calculate the extremum (minimum or maximum) of the integral

$$I = \int_{x_1}^{x_2} \phi(y(x), y'(x), x).$$

B. Example: Shortest distance between two points

What is the shortest distance between two points on a plane? We can draw a curve $y(x)$ connecting point 1 and point 2 and divide the curve into infinitesimally short ds pieces. The length of the curve is the sum of the short pieces:

$$I = \int_1^2 ds$$

For each infinitesimal piece

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'(x)^2} dx \tag{1}$$

so the functional is

$$\phi(y(x), y'(x), x) = \sqrt{1 + y'(x)^2}$$

C. Example: Shortest traveling time on a wire

Assume that a bead travels along a wire without friction in a uniform gravitational field (the gravitational field is parallel to the y axes). What shape of wire minimizes the time of travel from point 1 to point 2? The travel time is given by

$$I = \int_1^2 dt$$

where the travel time in an infinitesimal distance ds is $dt = ds/v$. From the energy conservation

$$\frac{1}{2}mv^2 = mgy$$

that is $v = \sqrt{2gy}$. Using this and eq.(1) we have

$$I = \int_1^2 \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy}} dx$$

and

$$\phi(y(x), y'(x), x) = \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy}}$$

D. Euler-Lagrange equations

Problem: Find the function $y(x)$ that makes

$$I = \int_{x_1}^{x_2} \phi(y(x), y'(x), x).$$

an extremum!

Solution: Let $y(x)$ the function that gives the extremum and

$$Y(x) = y(x) + \epsilon\eta(x)$$

an other function where $\eta(x)$ is an arbitrary function of x satisfying

$$\eta(x_1) = \eta(x_2) = 0$$

(all admissible curves go through the same endpoints) and ϵ is infinitesimally small number. The integral

$$I(\epsilon) = \int_{x_1}^{x_2} \phi(Y(x), Y'(x), x)dx$$

can be expanded in Taylor series around $\epsilon = 0$

$$I(\epsilon) = \int_{x_1}^{x_2} \phi(y(x), y'(x), x)dx + \int_{x_1}^{x_2} \left[\frac{\partial \phi}{\partial y}\eta(x) + \frac{\partial \phi}{\partial y'}\eta'(x)\right] dx,$$

where the $O(\epsilon^2)$ terms are omitted. $I(\epsilon)$ has an extrema at $\epsilon = 0$ if $\frac{dI(\epsilon)}{d\epsilon} = 0$,

$$\frac{dI(\epsilon}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial \phi}{\partial y}\eta(x) + \frac{\partial \phi}{\partial y'}\eta'(x)\right] dx = 0.$$

Integrating the second term by part

$$\int_{x_1}^{x_2} \frac{\partial \phi}{\partial y'}\eta'(x) = \frac{\partial \phi}{\partial y'}\eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial \phi}{\partial y'}\right)$$

where the first term is zero because $\eta(x)$ is zero at the endpoints. Using this relation, the condition of extremum is:

$$\int_{x_1}^{x_2} \eta(x)\left(\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y'}\right) dx = 0.$$

and because $\eta(x)$ is an arbitrary function, this equation implies that

$$\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y'} = 0.$$

(This is called the Euler-Lagrange equation.)

E. Shortest distance

The functional (from the example above is)

$$\phi(y(x), y'(x), x) = \sqrt{1 + y'(x)^2}.$$

The Euler-Lagrange equation is

$$\frac{d}{dx}\frac{\partial \phi}{\partial y'} = \frac{d}{dx}\frac{y'}{\sqrt{1 + y'(x)^2}} = \frac{y''}{(1 + y'(x)^2)^{3/2}} = 0$$

Because the denominator is positive the condition of extremum is $y'' = 0$, which means that the shortest path is $y = ax + b$ as expected.

V. HAMILTON'S PRINCIPLE (PRINCIPLE OF LEAST ACTION)

For every mechanical system there is a functional (called Lagrangian)

$$L(\underline{q}, \underline{\dot{q}}, t) = L(q_1, ..., q_s, \dot{q}_1, ..., \dot{q}_s, t)$$

and the system moves between two points in time (t_1 and t_2) in such a way that the *action*

$$S = \int_{t_1}^{t_2} L(\underline{q}, \underline{\dot{q}}, t)$$

has extremum (usually minimum).

From the calculus of variations we have learned that the action, S has extremum if the Euler-Lagrange equations are fulfilled:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \qquad (i = 1, ..., s)$$

The Lagrangian is defined only within an additive total time derivative of any function of coordinates and time. $L_1(\underline{q}, \underline{\dot{q}}, t)$ and

$$L_2(\underline{q}, \underline{\dot{q}}, t) = L_1(\underline{q}, \underline{\dot{q}}, t) + \frac{d}{dt} f(\underline{q}, t)$$

leads to the same equation of motion. To prove that we have to show that for

$$W = \frac{d}{dt} f(\underline{q}, t) = \sum_{i=1}^{s} \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$$

$$\frac{\partial W}{\partial q_i} - \frac{d}{dt}\frac{\partial W}{\partial \dot{q}_i} = 0.$$

First:

$$\frac{\partial W}{\partial q_i} = \sum_{j=1}^{s} \frac{\partial^2 f}{\partial q_j \partial q_i}\dot{q}_j + \frac{\partial^2 f}{\partial t \partial q_i}$$

Second:

$$\frac{d}{dt}\frac{\partial W}{\partial \dot{q}_i} = \frac{d}{dt}\frac{\partial f}{\partial q_i} = \sum_{j=1}^{s} \frac{\partial^2 f}{\partial q_j \partial q_i}\dot{q}_j + \frac{\partial^2 f}{\partial t \partial q_i}$$

proving the equivalency of the two Lagrangians.

A. Lagrangian of a free particle

The equation of motion of a free particle should be independent of the position of particle (homogeneity of the space), of the time (homogeneity of time) and the direction of velocity. The Lagrangian of a free particle can only depend upon the magnitude of velocity only:

$$L = L(\mathbf{v}^2)$$

The Lagrange equation for the free particle is then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = 0$$

which shows that

$$\frac{\partial L}{\partial v_i} = constant.$$

But $\frac{\partial L}{\partial v_i}$ is the function of the velocity so it can only be constant if

$$v_i = constant.$$

According to Galileo's relativity principle, the equations of motions in the inertial frame K and in the inertial frame K' moving with constant velocity with respect to K, should have the same form. First let us assume that K' moves with a constant infinitesimal velocity ϵ with respect to K. The velocity of the free particle in K is $\mathbf{v}$ and in K' is $\mathbf{v} + \epsilon$. The Lagrangians in the two systems can be related

$$L((\mathbf{v}+\epsilon)^2) = L(\mathbf{v}^2 + 2\mathbf{v}\cdot\epsilon + \epsilon^2) = L(\mathbf{v}^2) + \frac{\partial L}{\partial \mathbf{v}^2} 2\mathbf{v}\cdot\epsilon$$

(omitting the higher order terms. The last term is a total time derivative (and therefore leaves the equations of motions invariant) if

$$\frac{\partial L}{\partial \mathbf{v}^2} = constant.$$

This means that

$$L = constant \times \mathbf{v}^2 = \frac{1}{2}m\mathbf{v}^2$$