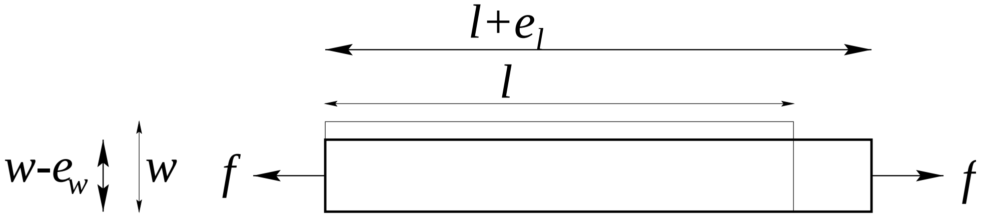


XXII. STRAIN AND STRESS

When an elastic body is deformed, internal restoring forces are produced. The **strain** at a point describes the deformation in the vicinity of that point. The **stress** describes the forces that maintain the deformation. In a *linear elastic body*, the stress is linearly related to the strain. Compressions, extensions, and shears are examples of strains. Pressure, tension, and shear stress are examples of stresses.

Strain ϵ is a fractional extension.

Stress τ is a force per unit area.

You may have encountered two quantities called the **Young's modulus** Y and the **Poisson ratio** σ of a material. These relate the strain and stress to each other as follows. If a rod of length l, width w, and cross–sectional area A is subjected to a force f and its length increases by e_l and its width *decreases* by e_w, then we define the Young's modulus to be the ratio of the lengthways stress f/A to the lengthways strain e_l/l,

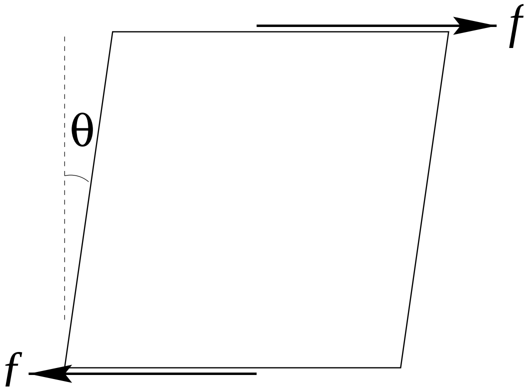
$$E = \frac{f/A}{e_l/l},$$

and we define the Poisson ratio to be

$$\nu = \frac{e_w/w}{e_l/l}.$$

Strains are dimensionless, so the Young's modulus has the dimensions of a stress. Example: The Young's modulus of steel is 2×10^{11} Pa.

The Poisson ratio is dimensionless. A typical value for σ is between 0 and 1/2; only values between -1 and 1/2 are physically possible, and negative values are rare — most things shrink widthways when you stretch them lengthways. Negative values are measured for special foams, etc. The Poisson ratio for cork is around zero (does not jam the bottle), the poisson ratio of rubber is almost 0.5 (incompressible).



You may also have come across the terms **bulk modulus** and **shear modulus**. The bulk modulus B is the ratio of the pressure to the fractional reduction in volume of a sample, when it is subjected to isotropic pressure.

$$K = \frac{p}{-\Delta V/V}.$$

The bulk modulus of steel is $160\ 10^9\ N/m^2$, that of water is $2.2\ 10^9\ N/m^2$. The shear modulus n is the ratio of the shear force per unit area to the shear angle, θ,

$$n = \frac{f/A}{\theta}.$$

A. What are strain and stress?

Strain and stress are not scalars. They both have directional properties. Strain and stress are not vectors either: vectors change sign if we rotate our coordinate system through 180 degrees, but a strain looks identical to itself if we spin through 180 degrees.

Strain and stress are matrices, also known as second-rank tensors.

B. Definition of strain

Imagine a deformation of a body such that the point in the body that was at location $\mathbf{x}$ is displaced through $\mathbf{u}(\mathbf{x})$ to $\mathbf{x} + \mathbf{u}(\mathbf{x})$. If the deformation is not a uniform translation or rotation then distances between pairs of points in the body will have changed. The **strain** at a point describes *by how much distances between nearby pairs of points have changed.*

Consider a pair of points $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$ in the undeformed body, where $d\mathbf{x}$ is a small displacement. These are deformed to locations $\mathbf{x} + \mathbf{u}(\mathbf{x})$ and $\mathbf{x} + d\mathbf{x} + \mathbf{u}(\mathbf{x} + d\mathbf{x})$. The squared distance between the deformed points is

$$\begin{aligned}
&\sum_i \left(dx_i + \sum_j \frac{\partial u_i}{\partial x_j} dx_j \right)^2 \\
&\quad = \sum_i (dx_i)^2 + 2 \sum_{i,j} \left(dx_i \frac{\partial u_i}{\partial x_j} dx_j \right) + \sum_{i,j,j'} \left(\frac{\partial u_i}{\partial x_j} dx_j \frac{\partial u_i}{\partial x_{j'}} dx_{j'} \right)
\end{aligned}$$

The first term, $\sum_i dx_i^2$, is the original squared distance between the points; the remaining terms give us the *change* in squared distance between the points, which is what we are interested in.

The change in squared distance, as a function of $d\mathbf{x}$, is a quadratic function of $d\mathbf{x}$. We can therefore manipulate it into quadratic form, and define the matrix in that quadratic form to be the strain tensor.

$$\begin{aligned}
D(d\mathbf{x}) &\equiv \sum_{i,j} dx_i \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dx_j + \sum_{i,j,j'} dx_j \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_{j'}} \right) dx_{j'} \\
&= \sum_{i,j} dx_i \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right] dx_j
\end{aligned}$$

We define the strain tensor to be

$$\epsilon_{ij} \equiv \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right],$$

then the change in squared distance is

$$D(d\mathbf{x}) = 2 \sum_{i,j} dx_i \, \epsilon_{ij} \, dx_j$$

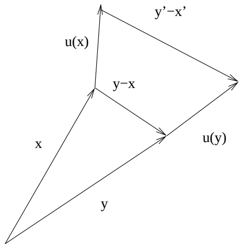
We will usually assume that we are concentrating on *small* strains, i. e., that the derivatives $\frac{\partial u_i}{\partial x_j}$ are all much smaller than 1. This assumption allows us to ignore the second term, which is quadratic in $\frac{\partial u_k}{\partial x_j}$, so that *the strain tensor for small distortions*is

$$\epsilon_{ij} \equiv \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right].$$

Alternative derivation: Consider two points $\mathbf{x}$ and $\mathbf{y}$ that moves, due to a deformation to

$$\mathbf{x}' = \mathbf{x} + \mathbf{u}(\mathbf{x}) \qquad \mathbf{y}' = \mathbf{y} + \mathbf{u}(\mathbf{y}).$$

This is too general, it includes translation ($\mathbf{u}$=const). We will instead use the vector



separation

$$\mathbf{y}' - \mathbf{x}' = \mathbf{y} - \mathbf{x} + \mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})$$

For small separation, we will use Taylor expansion up to linear order

$$(\mathbf{y}' - \mathbf{x}')_i - (\mathbf{y} - \mathbf{x})_i = \sum_{j=1}^{3} (\mathbf{y} - \mathbf{x})_j \frac{\partial u_i}{\partial x_j}$$

Now the matrix

$$\frac{\partial u_i}{\partial x_j}$$

can (always) be written as a sum of symmetric and antisymmetric components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \epsilon_{ij} + \Omega_{ij}$$

The second (antisymmetrix) part has three elements

$$\Omega_1 = \frac{\partial u_2}{\partial x_3} \qquad \Omega_2 = \frac{\partial u_3}{\partial x_1} \qquad \Omega_3 = \frac{\partial u_1}{\partial x_2}$$

and then we see that

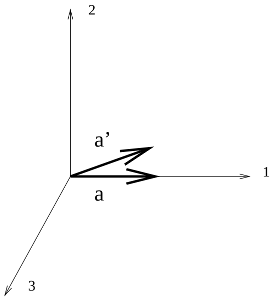
$$(\mathbf{y}' - \mathbf{x}')_i - (\mathbf{y} - \mathbf{x})_i = \sum_{j=1}^{3} \epsilon_{ij} (\mathbf{y} - \mathbf{x})_j + [\mathbf{\Omega} \times (\mathbf{y} - \mathbf{x})]_i \,.$$

The second part is an infinitesimal rigid rotation. The true internal deformation of elastic continuum thus arises only from the tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

called the elastic strain tensor.

$$[(\mathbf{y}' - \mathbf{x}')_i - (\mathbf{y} - \mathbf{x})_i]_{elastic} = \sum_{j=1}^{3} \epsilon_{ij} (\mathbf{y} - \mathbf{x})_j.$$



C. Examples of strain tensors

Suppose that $\mathbf{y}-\mathbf{x}=\mathbf{a}$ lies along the $\mathbf{e}_1$ axis of our coordinate system,$\mathbf{a}=(a,0,0)$. Then the elastic deformation is

$$\mathbf{a}'-\mathbf{a}=\epsilon_{11}a\mathbf{e}_1+\epsilon_{21}a\mathbf{e}_2+\epsilon_{31}a\mathbf{e}_3$$

that is

$$\mathbf{a}'=(1+\epsilon_{11})a\mathbf{e}_1+\epsilon_{21}a\mathbf{e}_2+\epsilon_{31}a\mathbf{e}_3.$$

The length of $\mathbf{a}$ changed as

$$a'=\sqrt{|\mathbf{a}'\cdot\mathbf{a}'|}\approx a(1+\epsilon_{11})$$

which shows that

$$\epsilon_{11}=\frac{a'-a}{a}$$

is the fractional change of lenght. The direction of $\mathbf{a}'$ is also changed. Now let's assume that there is an other vector $\mathbf{b}$ which originally orthogonal to $\mathbf{a}$, and lies in the $\mathbf{e}_2$ direction, $\mathbf{b}=(0,b,0)$, under deformation

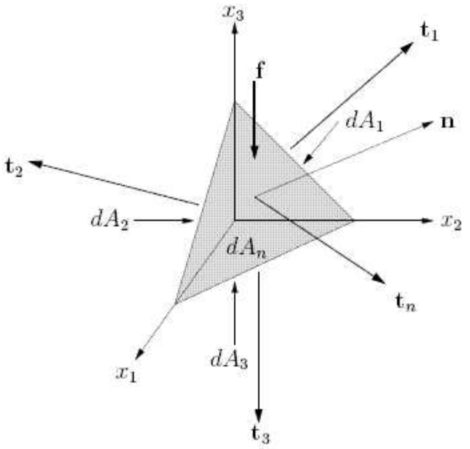
$$\mathbf{b}'=\epsilon_{12}b\mathbf{e}_1+(1+\epsilon_{22})b\mathbf{e}_2+\epsilon_{32}b\mathbf{e}_3.$$

The angle between the two vectors after deformation

$$\mathbf{a}'\cdot\mathbf{b}'\approx ab(\epsilon_{12}+\epsilon_{21})$$

Similarly, under elastic deformation, the volume $V=abc$ bounded by three orthogonal vector $a\mathbf{e}_1$,$b\mathbf{e}_2$ and $c\mathbf{e}_3$

$$\begin{aligned} V' &= \mathbf{a}'\cdot\mathbf{b}'\times\mathbf{c}'=\det\begin{pmatrix} a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \\ c_1' & c_2' & c_3' \end{pmatrix} \\ &= abc(1+\epsilon_{11}+\epsilon_{22}+\epsilon_{33})=V(1+\mathrm{Tr}\boldsymbol{\epsilon}). \end{aligned}$$



This can be rewritten as

$$\mathrm{Tr}\boldsymbol{\epsilon} = \nabla\mathbf{u} = \frac{V' - V}{V} = \frac{dV}{V} = -\frac{d\rho}{\rho}$$

D. The stress tensor

Conservation of mass:

$$\frac{d}{dt}\int_V \rho(\mathbf{r}, t)dV$$

Stress

$$\boldsymbol{\sigma}(\mathbf{r}, \mathbf{n}) = \lim_{A \to \mathbf{r}} \frac{\mathbf{F}(A)}{A}$$

where $\mathbf{F}(A)$ is a surface force (force acting on the surface) and $\mathbf{n}$ is the normal vector of the surface at point $\mathbf{r}$. Reminder: A surface can be parametrized by two coordinates, e.g. u_1, u_2, that is $\mathbf{r}(u_1, u_2)$ then the normal vector is a unit vector parallel to

$$\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}.$$

The surface force can be expressed by the stress as

$$\mathbf{F}(A) = \int_A \boldsymbol{\sigma}(\mathbf{r}, \mathbf{n})dA$$

The volume/body force ($\mathbf{f}$ is the volume/body force density,independent of the neighborhood, e.g. gravitational force)

$$\mathbf{F}(V) = \int_V \mathbf{f}\rho dV$$

Momentum density: The surface force can be expressed by the stress as

$$\mathbf{p} = \rho\mathbf{v}$$

Momentum

$$\mathbf{P} = \int \rho \mathbf{v} dV$$

Newton's second law

$$\dot{\mathbf{P}} = \mathbf{F}(A) + \mathbf{F}(V)$$

$$\frac{d}{dt}\int_V \mathbf{p} dV = \int_V \rho \mathbf{f} dV + \oint_A \boldsymbol{\sigma}(\mathbf{r}, \mathbf{n}) dA$$

An important relation:

$$\boldsymbol{\sigma}(\mathbf{r}, \mathbf{n}) = -\boldsymbol{\sigma}(\mathbf{r}, -\mathbf{n})$$

Torque

$$\boldsymbol{\Gamma}(V) = \int_V \rho(\mathbf{r} \times \mathbf{f}) dV$$

$$\boldsymbol{\Gamma}(A) = \int_A (\mathbf{r} \times \boldsymbol{\sigma}) dA$$

Angular momentum density

$$\mathbf{l} = \mathbf{r} \times \mathbf{f}$$

Angular momentum

$$\mathbf{L} = \int_V \mathbf{l} dV$$

and the equation of motion

$$\dot{\mathbf{L}} = \mathbf{M}(A) + \mathbf{M}(V)$$

or in detail

$$\frac{d}{dt}\int_V (\mathbf{r} \times \mathbf{p} dV = \int_V \rho(\mathbf{r} \times \mathbf{f}) dV + \oint_A (\mathbf{r} \times \boldsymbol{\sigma}) dA$$

Static equations:

$$\int_V \rho \mathbf{f} dV + \oint_A \boldsymbol{\sigma}(\mathbf{r}, \mathbf{n}) dA = 0$$

$$\int_V \rho(\mathbf{r} \times \mathbf{f}) dV + \oint_A (\mathbf{r} \times \boldsymbol{\sigma}) dA = 0$$

From these equations the condition of equilibrium for the tetrahedron on the figure is

$$\int_V \rho \mathbf{f} dV + \int_{A_n} \boldsymbol{\sigma}(\mathbf{r}, \mathbf{n}) dA_n - \sum_{i=1}^{3} \int_{A_i} \boldsymbol{\sigma}(\mathbf{r}, \mathbf{e}_i) dA_i = 0$$

where

$$dA_i = dA_n \cos(\mathbf{n}, \mathbf{e}_i) = dA_n \mathbf{n} \cdot \mathbf{e}_i$$

and with this

$$\int_V \rho \mathbf{f} dV + \int_{A_n} \left(\boldsymbol{\sigma}(\mathbf{r}, \mathbf{n}) - \sum_{i=1}^{3} \boldsymbol{\sigma}(\mathbf{r}, \mathbf{e}_i) \mathbf{n} \cdot \mathbf{e}_i \right) dA_n = 0$$

taking the $V \to 0$ limit, the volume integral will be equal to zero, and the integrand of the surface integral should also go to zero, that is

$$\boldsymbol{\sigma}(\mathbf{r}, \mathbf{n}) = \sum_{i=1}^{3} \boldsymbol{\sigma}(\mathbf{r}, \mathbf{e}_i) \mathbf{n} \cdot \mathbf{e}_i = \sum_{i=1}^{3} \boldsymbol{\sigma}(\mathbf{r}, \mathbf{e}_i) n_i$$

where $\mathbf{n} = (n_1, n_2, n_3)$. In matrix form

$$\begin{pmatrix} \boldsymbol{\sigma}_1(\mathbf{r}, \mathbf{n}) \\ \boldsymbol{\sigma}_2(\mathbf{r}, \mathbf{n}) \\ \boldsymbol{\sigma}_3(\mathbf{r}, \mathbf{n}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_1(\mathbf{r}, \mathbf{e}_1) & \boldsymbol{\sigma}_1(\mathbf{r}, \mathbf{e}_2) & \boldsymbol{\sigma}_1(\mathbf{r}, \mathbf{e}_3) \\ \boldsymbol{\sigma}_2(\mathbf{r}, \mathbf{e}_1) & \boldsymbol{\sigma}_2(\mathbf{r}, \mathbf{e}_2) & \boldsymbol{\sigma}_2(\mathbf{r}, \mathbf{e}_3) \\ \boldsymbol{\sigma}_3(\mathbf{r}, \mathbf{e}_1) & \boldsymbol{\sigma}_3(\mathbf{r}, \mathbf{e}_2) & \boldsymbol{\sigma}_3(\mathbf{r}, \mathbf{e}_3) \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

or

$$\boldsymbol{\sigma}(\mathbf{r}, \mathbf{n}) = \hat{\sigma} \mathbf{n}$$

where

$$\hat{\sigma}_{ij} = \mathbf{e}_i \cdot \boldsymbol{\sigma}(\mathbf{r}, \mathbf{e}_j)$$

is the stress tensor.

Example:

$$\boldsymbol{\sigma}(\mathbf{r}, \mathbf{e}_3) = \boldsymbol{\tau} + \hat{\sigma}_{33} \mathbf{e}_3 \qquad \boldsymbol{\tau} = \hat{\sigma}_{13} \mathbf{e}_1 + \hat{\sigma}_{23} \mathbf{e}_2$$

and σ_{33} positive : tension; σ_{33} negative : compression, $\boldsymbol{\tau}$ shear stress.

The static equation now can be written as

$$\int_V \rho \mathbf{f} dV + \oint_A \hat{\sigma} \mathbf{dA} = 0 \qquad \mathbf{dA} = \mathbf{n} dA$$

and using Gauss's theorem:

$$\oint_A \hat{\sigma} \mathbf{dA} = \int_V \nabla \cdot \hat{\sigma} dV$$

we have

$$\int_V (\rho \mathbf{f} + \nabla \cdot \hat{\sigma}) \, dV = 0$$

that is

$$\rho \mathbf{f} + \nabla \cdot \hat{\sigma} = 0$$

which is in detail reads as

$$\rho f_i + \frac{\partial \sigma_{i1}}{\partial x} + \frac{\partial \sigma_{i2}}{\partial y} + \frac{\partial \sigma_{i3}}{\partial z} = 0$$

$$\int_V \rho \mathbf{f} dV + \oint_A \hat{\sigma} \mathbf{dA} = 0 \qquad \mathbf{dA} = \mathbf{n} dA$$

For the torque

$$\int_V \rho (\mathbf{r} \times \mathbf{f}) dV + \oint_A (\mathbf{r} \times \hat{\sigma} \mathbf{n}) dA = 0$$

in a similar way, by using Gauss's divergence theorem

$$\int_V \left(\rho (\mathbf{r} \times \mathbf{f}) + \nabla \cdot \hat{\lambda} \right) dV = 0$$

that is

$$\rho \mathbf{r} \times \mathbf{f} + \nabla \cdot \hat{\lambda} = 0 \tag{1}$$

where

$$\hat{\lambda} = \mathbf{r} \times \hat{\sigma}.$$

Some details: The equation for the angular momentum can be written in components in a more detailed way as

$$\sum_{j,k=1}^{3} \int_V \epsilon_{ijk} x_j \rho f_k dV + \sum_{j,k,l=1}^{3} \oint_A \epsilon_{ijk} x_j \sigma_{lk} n_l dA = \sum_{j,k=1}^{3} \int_V \rho \epsilon_{ijk} x_j \dot{u}_k dV \qquad (i = 1, 2, 3)$$

where

$$\mathbf{n} = (n_1, n_2, n_3) \qquad \mathbf{f} = (f_1, f_2, f_3) \qquad \mathbf{r} = (x_1, x_2, x_3)$$

and the totally antisymmetric tensor ϵ_{ijk} is defined as

$$\epsilon_{ijk} = \begin{cases} 1 \text{ if } i, j, k \text{ even permutation i.e., } 123, 231, 312 \\ -1 \text{ if } i, j, k \text{ odd permutation i.e., } 321, 213, 132 \\ 0 \text{ if } i, j, k \text{ if any two indeces are the same e.g. } 221, 222 \end{cases}$$

Using this notation, the component of a vector product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$

$$c_i = \sum_{j,k=1}^{3} \epsilon_{ijk} a_j b_k$$

The divergence theorem for a second rank tensor t_{ij} in components reads

$$\sum_{j=1}^{3} \int_V \frac{\partial t_{ji}}{x_j} dV = \sum_{j=1}^{3} \oint_A t_{ji} n_j dA.$$

Applying this for $\epsilon_{ijk} x_j \sigma_{lk}$ in the equation of the angular momentum one has

$$\sum_{j,k=1}^{3} \int_V [\epsilon_{ijk} (x_j \rho f_k + \sum_{l=1}^{3} \frac{\partial x_j \sigma_{lk}}{\partial x_l})] dV = \sum_{j,k=1}^{3} \int_V \epsilon_{ijk} x_j \rho \dot{u}_k dV \qquad (i = 1, 2, 3)$$

which in local form is

$$\sum_{j,k=1}^{3}[\epsilon_{ijk}x_j f_k+\sum_{l=1}^{3}\frac{\partial x_j\sigma_{lk}}{\partial x_l})]=\sum_{j,k=1}^{3}\epsilon_{ijk}x_j\dot{u}_k\qquad(i=1,2,3)$$

If $\dot{u}_k=0$ then

$$\sum_{j,k=1}^{3}[\epsilon_{ijk}(x_j f_k+\sum_{l=1}^{3}\frac{\partial x_j\sigma_{lk}}{\partial x_l})]=0\qquad(i=1,2,3)$$

which is identical to eq. (1). Using these equations one can easily show that the stress tensor is symmetric:

$$\hat{\sigma}_{ij}=\hat{\sigma}_{ji}.$$

E. Hook's law

Hook's law relates the strain to the stress. The most elementary one dimensional case

$$\sigma_{11}=E\epsilon_{11}$$

where E is called the modulus of elasticity (Young modulus). The most general linear connection would be

$$\sigma_{ij}=\sum_{kl}Y_{ij,kl}\epsilon_{kl}$$

where $Y_{ij,kl}$ could have a 9x9 elements, but the strain tensor and the stress tensor are both symmetric, so they only have 6 independent elements, the elements of $Y_{ij,kl}$ are then 6x6. Varios symmetry properties also reduce the number of independent elements of $Y_{ij,kl}$. We restric ourself to *isotropic* materials, in which case there are two independent elements λ and μ the Lame coefficients and the relation between the strain and stress is

$$\hat{\sigma}=2\mu\hat{\epsilon}+\lambda\mathrm{Tr}(\hat{\epsilon})\hat{1}$$

or the "inverse" relation

$$\hat{\epsilon}=\frac{1}{2\mu}\hat{\sigma}-\frac{\lambda}{2\mu(2\mu+3\lambda}\mathrm{Tr}(\hat{\sigma})\hat{1}$$

We will show that using the Lame coefficients, the Young modulus

$$E=\frac{\mu(2\mu+3\lambda)}{\mu+\lambda}$$

the Poisson's ratio

$$\nu=\frac{\lambda}{2(\mu+\lambda)}$$

(so it is really limited between -1 and 1/2), the bulk modulus

$$K = \frac{2\mu + 3\lambda)}{3}$$

F. Equations governing the displacement

The equilibrium displacements are determined by

$$\rho \mathbf{f} + \nabla \cdot \hat{\sigma} = 0, \tag{2}$$

$$\hat{\sigma} = 2\mu\hat{\epsilon} + \lambda \mathrm{Tr}(\hat{\epsilon})\hat{1} \tag{3}$$

$$\epsilon_{ij} \equiv \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]. \tag{4}$$

Goal: given $\mathbf{f}$ and the material parameters determine the displacement $\mathbf{u}(\mathbf{r})$ from these partial differential equations. These three equations can be combined and one gets the

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \rho\mathbf{f} = 0 \tag{5}$$

Lame-Navier equations.