I. KINEMATICS: TRANSFORMATION OF COORDINATE SYSTEMS

Let $\mathbf{e}_i$ and $\mathbf{e}'_i$ $(i = 1, \ldots 3)$ two orthonormal basis sets defining two coordinate systems. We allow $\mathbf{e}'_i$ to be time dependent, but $\mathbf{e}_i$ is fixed in time:

$$\dot{\mathbf{e}}_i = \frac{d\mathbf{e}_i}{dt} = 0.$$

A position vector $\mathbf{r}$ can be represented in both coordinate systems as

$$\mathbf{r} = \sum_{i=1}^{3} r_i \mathbf{e}_i = \sum_{j=1}^{3} r'_j \mathbf{e}'_j,$$

the components in the two representations will be denoted as

$$\underline{r} = (r_1, r_2, r_3) \qquad \underline{r'} = (r'_1, r'_2, r'_3)$$

The components are not independent

$$r_i = \sum_{j=1}^{3} r'_j \mathbf{e}_i \cdot \mathbf{e}'_j = \sum_{j=1}^{3} C_{ij} r'_j,$$

where the basis transformation matrix is

$$C_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$$

and

$$\mathbf{e}'_j = \sum_{i=1}^{3} C_{ij} \mathbf{e}_i \qquad \underline{r} = C \underline{r'}$$

Due to the ortonormality of the basis

$$C^{-1} = C^T \qquad \det C = \pm 1.$$

The time derivative of the $\mathbf{e}'_j$ basis vectors can be expressed as a linear combination of the $\mathbf{e}'_j$ basis vectors:

$$\dot{\mathbf{e}}'_j = \sum_{i=1}^{3} F_{ij} \mathbf{e}'_i$$

where

$$F_{ij} = \mathbf{e}'_i \cdot \dot{\mathbf{e}}'_j$$

By calculating the time derivative of

$$\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$$

one obtains

$$0 = \dot{\mathbf{e}}_i' \cdot \mathbf{e}_j' + \mathbf{e}_i' \cdot \dot{\mathbf{e}}_j' = F_{ij} + F_{ji}$$

which means that F_{ij} is antisymmetric

$$F_{ij} = -F_{ji}.$$

A 3x3 antisymmetric matrix has three independent elements (the diagonal must be zero and only the lower or upper triangle is filled with independent elements), so we can write it in the form:

$$F = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

The time derivative of the position vector in the two coordinate systems is represented by the components

$$\underline{v} = (v_1, v_2, v_3) \qquad \underline{v'} = (v_1', v_2', v_3')$$

In the $\mathbf{e}_i$ system

$$\mathbf{v} = \dot{\mathbf{r}} = \sum_{i=1}^{3} \dot{r}_i \mathbf{e}_i \quad \text{that is} \quad \underline{v} = \underline{\dot{r}}$$

In the $\mathbf{e}_i'$ system

$$\dot{\mathbf{r}} = \sum_{i=1}^{3} \dot{r}_i' \mathbf{e}_i' + \sum_{i=1}^{3} r_i' \dot{\mathbf{e}}_i' \quad \text{that is} \quad \underline{v'} = \underline{\dot{r}'} + F\underline{r'}.$$

The last expression can be written in a more elegant form:

$$\underline{v'} = \underline{\dot{r}'} + \underline{\omega} \times \underline{r'}, \qquad \underline{\omega} = (\omega_1, \omega_2, \omega_3).$$

Using this equation, one can easily calculate $\dot{\mathbf{v}}$

$$\underline{a'} = \underline{\dot{v}'} + \underline{\omega} \times \underline{v'} = \underline{\ddot{r}'} + 2\underline{\omega} \times \underline{\dot{r}'} + \underline{\omega} \times (\underline{\omega} \times \underline{r'}) + \underline{\dot{\omega}} \times \underline{r'}$$

A. Example: Cartesian coordinate system

$$\begin{aligned} \mathbf{e}_x &= (1, 0, 0) \\ \mathbf{e}_y &= (0, 1, 0) \\ \mathbf{e}_z &= (0, 0, 1) \end{aligned}$$

B. Example: Polar coordinate system

Basis vectors:

$$\begin{aligned}
\mathbf{e}_r &= \cos\phi \mathbf{e}_x + \sin\phi \mathbf{e}_y \\
\mathbf{e}_\phi &= -\sin\phi \mathbf{e}_x + \cos\phi \mathbf{e}_y \\
\mathbf{e}_z &= \mathbf{e}_z
\end{aligned}$$

$$\mathbf{r} = r\mathbf{e}_r \quad \underline{r'} = (r, 0, 0)$$

Basis transformation

$$C = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$F = \begin{pmatrix} 0 & -\dot\phi & 0 \\ \dot\phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\underline{\omega} = (0, 0, \dot\phi)$$

$$\underline{v'} = (\dot r, r\dot\phi, 0) \qquad \underline{a'} = (\ddot r - r\dot\phi^2, 2\dot r\dot\phi + r\ddot\phi, 0)$$

C. Example: Spherical polar coordinate system

Basis vectors:

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta\cos\phi \mathbf{e}_x + \sin\theta\sin\phi \mathbf{e}_y + \cos\theta \mathbf{e}_z \\
\mathbf{e}_\theta &= \cos\theta\cos\phi \mathbf{e}_x + \cos\theta\sin\phi \mathbf{e}_y - \sin\theta \mathbf{e}_z \\
\mathbf{e}_\phi &= -\sin\phi \mathbf{e}_x + \cos\phi \mathbf{e}_y
\end{aligned}$$

$$\mathbf{r} = r\mathbf{e}_r \quad \underline{r'} = (r, 0, 0)$$

basis transformation

$$C = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}.$$

$$\underline{\omega} = (\dot{\phi}\cos\theta, -\dot{\phi}\sin\theta, \dot{\theta})$$

$$\underline{v'} = (\dot{r}, r\dot{\theta}, r\dot{\phi}\sin\theta)$$

$$\underline{a'} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta, 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta, 2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)$$

II. DYNAMICS: BASIC PRINCIPLES

A. Newton's laws

(i) In the inertial frame, every body remains at rest or in a uniform motion unless acted on by a force **F**.

$$\mathbf{F} = 0 \Rightarrow \mathbf{p} = constant$$

(ii) Application of a force alters the momentum

$$\mathbf{F} = \dot{\mathbf{p}}$$

(iii) To each action there is an equal and opposite reaction

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

(iv) Superposition of forces

$$\mathbf{F} = \sum_i \mathbf{F}_i$$

B. Conservation laws

Linear momentum conservation

$$\mathbf{F} = 0 \Rightarrow \mathbf{p} = constant$$

Angular momentum conservation

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \mathbf{F} = \Gamma$$

(Γ is the torque).

Beware!

$$\mathbf{L} = (\mathbf{r} - \mathbf{r}_0) \times \mathbf{p},$$

where $\mathbf{r}_0$ is the origin of the reference frame. **L** depends on the choice of the coordinate system.

Work:

$$W_{1\to 2} = \int_1^2 \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

the integral is along a line: $\mathbf{r} = \mathbf{r}(\sigma)$

$$W_{1\to 2} = \int_1^2 \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{d\sigma} d\sigma$$

Conservative forces:

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}),$$

(where U is called potential). Alternative definitions:

$$\nabla \times \mathbf{F}(\mathbf{r}) = 0$$

$$\int_1^2 \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0 \qquad \text{for all closed paths}$$

Work in case of conservative forces:

$$W_{1\to 2} = -\int_1^2 \nabla U(\mathbf{r}) \cdot d\mathbf{r} = -\int_1^2 dU(\mathbf{r}) = U_1 - U_2$$

Kinetic energy

$$T = \frac{1}{2} m \mathbf{v}^2$$

Time derivative of the kinetic energy

$$\dot{T} = m\mathbf{v}\dot{\mathbf{v}} = \mathbf{F}\mathbf{v}$$

Energy conservation: By integrating this equation between times t_1 and t_2

$$W_{1\to 2} = \frac{1}{2} m \mathbf{v}_1^2 - \frac{1}{2} m \mathbf{v}_2^2 \qquad \mathbf{v}_1 = \mathbf{v}(t_1) \qquad \mathbf{v}_2 = \mathbf{v}(t_2).$$